Optimal Price of Entry into a Competition

Ginzburg, Boris

Department of Economics, Universidad Carlos III de Madrid

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Boris Ginzburg†

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Abstract

A continuum of contestants are choosing whether to enter a competition. Each contestant has a type, and of those who enter, the ones with highest types receive prizes. A profit-maximising firm controls entry, and charges a price for it. I analyse how the mass and value of prizes determine equilibrium price and intensity of entry. An increase in the value of each prize leads the firm to increase the price while keeping intensity of entry fixed. Conversely, when the mass of prizes increases, the firm initially keeps the price constant while entry increases; and later – raises the price.

Keywords: contests, entry

JEL codes: C72, D82

1 Introduction

Access to various competitive settings is often controlled by gatekeepers, who require contestants to pay for entry. For example, prospective students who consider applying for university admission or for competitive scholarships may need to pay a firm to take a GRE test in order to apply. Artists or athletes planning to join a competition are often required to pay for membership in an association. Similarly, academics who submit papers for a conference

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†Department of Economics, Universidad Carlos III de Madrid, Calle Madrid 126, 28903 Getafe, Spain. Email: bginzbur@eco.uc3m.es. Website: `sites.google.com/site/borgin/`.

with a fixed number of slots are sometimes asked to join a professional society, for a fee, as a condition for applying. If the gatekeeper is interested in maximising its profit from entry, what entry price will it set, what proportion of contestants will apply, and what are the effects of a change in the level of competitiveness and the value of winning?

To answer these questions, this paper models a continuum of contestants who choose whether to enter a competition for a continuum of prizes. Each contestant has a type, and out of those who enter the competition, contestants with the highest types each get one prize. Each contestant is privately informed about her type, but does not know the total mass of contestants, which is drawn from a continuous distribution – thus, a contestant is uncertain about the number of her competitors who have higher types. The key feature of the model is that entry into the competition is controlled by a profit-maximising firm, who charges a price for entering.

The paper shows how the mass of prizes and the value of each prize affect the optimal entry price and the proportion of contestants who enter the competition. When the mass of prizes is higher than some cutoff, the firm sets a price at which all contestants enter. Otherwise, the price is such that some contestants do not enter.

I then consider the effects of a change in the mass of prizes, as well as in the value of winning the prize. If the entry price was exogenously fixed, an increase in the mass of prizes or in the value of each prize would raise the expected payoff from entering the competition. Hence, more contestants would enter.

However, since the price of entry is endogenously determined, the situation is different. A change in the value of the prize induces the firm to increase the price of entry. I show that such an increase fully absorbs the effect of an increase in the value of the prize. Hence, the intensity of entry is unaffected.

On the other hand, the effect of a change in the mass of prizes depends on how large the mass initially is. If the mass of prizes is below a certain level, the firm reacts to an increase in the mass by allowing the intensity of entry to rise while keeping the price of entry unchanged. When the mass of prizes becomes large enough, inducing full entry becomes optimal for the firm. After that, a further increase in the mass of prizes leads the firm to increase the price.

This paper adds to the literature on entry into competitive settings, such as auctions (McAfee and McMillan, 1987; Levin and Smith, 1994; Arozamena and Weinschelbaum, 2011; Moreno and Wooders, 2011; Li, 2017; Lee and Li, 2019) or contests (Fu and Lu, 2010; Kaplan and Sela, 2010; Fu et al., 2015). In particular, Morgan et al. (2017) study entry into contests with a continuum of agents and prizes; while Ginzburg (2019) analyses a costly competitive test in which, as in this paper, winning depends on the contestant's exogenous type, rather than on her endogenously chosen effort or bid. The key difference of this paper is that the cost of entry is not exogenously fixed, but is set by a firm. The firm maximises its profit from entry, and has no stake in the outcome of the competition, unlike an auctioneer or a contest designer.

In addition, a number of studies have looked at contests with a stochastic number of players (Myerson and Wärneryd, 2006; Münster, 2006; Lim and Matros, 2009; Fu et al., 2011; Kahana and Klunover, 2015, 2016; Boosey et al., 2017, 2019; Gu et al., 2019). In that literature, nature exogenously draws the number of players, who then all participate in a contest[1]. In this paper too the mass of contestants is drawn by nature; however, the mass of contestants who actually participate in the contest is endogenous. Specifically, each contestant faces an entry cost (endogenously set by a firm), and decides whether to enter.

The rest of the paper is structured as follows. Section 2 introduces the model. Section 3 characterises equilibrium price and entry level. Section 4 describes the effects of a change in the value and mass of prizes on the price and intensity of entry. Section 5 discusses the role of the assumptions on the information available to the firm, and on the shape of the distribution of the mass of contestants; it then shows that the main results remain largely unchanged if these assumptions are relaxed. Finally, Section 6 concludes. All proofs are in the appendix.

2 Model

There is a continuum of contestants, and a continuum of prizes. The value of a prize to each contestant is $v > 0$. The mass of prizes is $m > 0$. The mass of contestants $y \in (0, +\infty)$ is drawn from a smooth distribution G with density g and full support. I will assume that G is unimodal with mode k – that is, G is strictly convex on $(0, k)$, and strictly concave on $(k, +\infty)$ for

[1] Gu et al. (2019) also consider endogenous entry, but at a fixed, rather than endogenously determined, cost.

some $k \in (0, +\infty)$.[2]

Each contestant i has a type $\theta_i \in [0,1]$, drawn from a distribution F with full support. Each contestant knows her type, but not the types of other contestants or the mass of contestants. Each contestant decides whether to enter the competition for prizes. Entry is controlled by a firm. The firm selects a price p that each contestant needs to pay for entry.

The timing is as follows. First, the firm selects p.[3] Then, nature draws y and the type of each contestant. Each contestant learns his type. Contestants then simultaneously decide whether to enter the competition. The strategy of each contestant is $h : [0,1] \to [0,1]$, such that a contestant with type θ enters the competition with probability $h(\theta)$.

Contestants who do not enter receive a payoff of 0. Contestants who enter pay the price p. If the mass of contestants who enter is greater than m, then out of contestants who enter, mass m of those who have the highest types receive one prize each. Formally, if the set of contestants who enter is S, then each contestant from the set $S \cap \left[\tilde{\theta}, 1\right]$ receives one prize, while other contestants do not receive prizes, where $\tilde{\theta}$ is a type such that the mass of $S \cap \left[\tilde{\theta}, 1\right]$ equals m. On the other hand, if the mass of contestants who enter is smaller than 1, then each contestant who enters receives a prize. The payoff of the firm is $p\mu_S$, where μ_S is the mass of S.

3 Equilibrium

If $p > v$, no contestant enters, and the firm receives zero profit. The firm is thus better off setting some price $p \in (0, v]$, and hence every equilibrium will be of this type.

We can now show that every equilibrium is of a cutoff type, as the following result describes:

Lemma 1. *Every equilibrium is characterised by a cutoff $\hat{\theta}$ given by*

$$vG\left[\frac{m}{1 - F\left(\hat{\theta}\right)}\right] - p = 0 \tag{1}$$

[2]Section 5 discusses the case when G has a generic shape.

[3]Thus, the firm is assumed to have the same information as the contestants. Section 5 discusses an alternative setup in which the firm receives some information about y before selecting p.

such that all contestants with types above $\hat{\theta}$ enter with certainty, and the mass of contestants whose types are below $\hat{\theta}$ and who enter is zero.

Intuitively, the probability that a contestant receives the prize is increasing in her type. Hence, any equilibrium is characterised by a cutoff $\hat{\theta}$ such that a contestant enters if and only if her type is above $\hat{\theta}$. The share of contestants that enter is thus $1 - F\left(\hat{\theta}\right)$. If a contestant with type $\hat{\theta}$ enters, she pays p, and receives a prize worth v if and only if $y\left[1 - F\left(\hat{\theta}\right)\right] \leq m$. The probability of this event is $G\left[\frac{m}{1-F\left(\hat{\theta}\right)}\right]$. If she does not enter, her payoff is zero. At the equilibrium, she must be indifferent between entering and not entering.

At the same time, the lemma allows for a zero-mass set of contestants with types below $\hat{\theta}$ to enter at the equilibrium[4]. However, from the firm's point of view, this possibility is largely irrelevant, since entry of a zero-measure set of contestants does not affect the firm's profit. Hence, for simplicity I will focus on equilibria in which no contestants with types below $\hat{\theta}$ enter the competition.

Let $x \equiv \frac{1}{1-F\left(\hat{\theta}\right)} \in [1, +\infty)$. Then Lemma 1 implies that at the equilibrium we have

$$p = vG\left(mx\right) \tag{2}$$

The mass of contestants who enter equals $y\left[1 - F\left(\hat{\theta}\right)\right] = \frac{y}{x}$, and the firm receives p from each of them. The firm's problem then is

$$\max_{p \in [0,v]} \mathrm{E}\left[\frac{y}{x}p\right] \text{ subject to } p = vG\left(mx\right)$$

Since there is a one-to-one relationship between x and p, the firm's problem can be written as that of selecting the optimal x. A larger value of x corresponds to a lower value of $1 - F\left(\hat{\theta}\right)$, and hence to lower intensity of entry. Let x^* be the firm's equilibrium choice of x. The firm's problem can

[4] The reason for this is that since the set of contestants is a continuum, a deviation by a single contestant with type below $\hat{\theta}$ does not change the probability of any contestant winning the prize. Hence, if a contestant with type below $\hat{\theta}$ enters the competition, her chance of winning a prize, and hence her payoff, would be the same as that of a contestant with type $\hat{\theta}$, who is indifferent between entering and not entering.

then be rewritten as

$$\max_{x\in[1,+\infty)} \frac{vG(mx)}{x}\mathrm{E}[y] \tag{3}$$

This is equivalent to maximising $\frac{G(mx)}{x}$, that is, to maximising the average value of the function $G(mx)$. Depending on the shape of G, (3) can either have an interior solution with $x^* > 1$, or a corner solution with $x^* = 1$. In the latter case, $F\left(\hat{\theta}\right) = 0$, that is, the firm selects a price that ensures full entry. To distinguish between these cases, the following technical result will be useful:

Lemma 2. *There exists a unique r such that $G(x) - xg(x) < 0$ if and only if $x < r$. Furthermore, $r > k$.*

Thus, $G(x) - xg(x) = 0$ at $x = r$. Hence, $g(r) = \frac{G(r)}{r}$ – that is, r is a point at which the slope of $G(r)$ equals the average value of $G(r)$. By Lemma 2, this point is to the right of k, and thus lies on the concave part of G.

Using Lemma 2, we can characterise the firm's equilibrium strategy as follows:

Proposition 1. *If $m \geq r$, then $x^* = 1$. If $m < r$, then $x^* = \frac{r}{m} > 1$.*

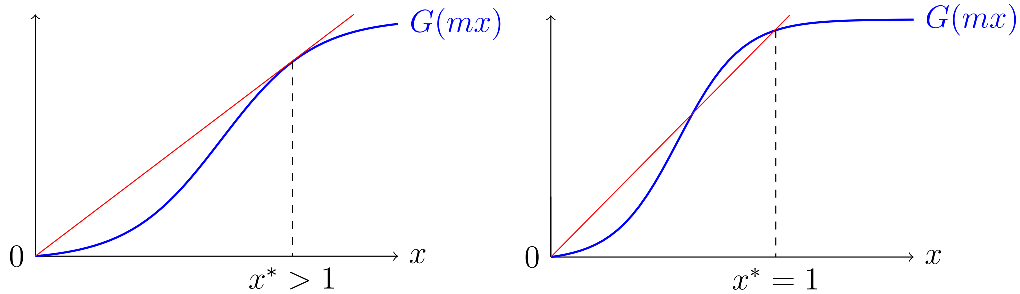
Proposition 1 says that the shape of the equilibrium depends on how large the mass of prizes is. If the mass of prizes is greater than r, the equilibrium is such that $x^* = 1$ – that is, all contestants enter. The firm then selects the largest price under which full entry is achieved. By (2), that price equals $vG(m)$.

On the other hand, if the mass of prizes is below r, the firm selects a price at which $x^* = \frac{r}{m} > 1$. Hence, contestants with sufficiently low types do not enter. The price then equals $vG(mx^*) = vG(r)$. Lemma 2 implies that in this case, x^* is given by

$$mx^* g(mx^*) = G(mx^*) \tag{4}$$

Additionally, note that by Lemma 2, in both cases, $mx^* \geq r > k$, and hence $G(\cdot)$ is concave at mx^*.

The two cases are illustrated in Figure 1. Panel (a) shows the case with partial entry, when $m < r$. In this case, x^* is a value of x at which $mg(mx) = \frac{G(mx)}{x}$, that is, at which the slope of $G(mx)$ equals the average value of



(a) Equilibrium with partial entry.

(b) Equilibrium with full entry.

Figure 1: Firm's equilibrium strategy.

$G(mx)$. Geometrically, x^* is a point such that the line connecting the point $(x^*, G(mx^*))$ to the point $(0,0)$ is tangent to $G(mx^*)$.

At the same time, panel (b) shows the case with full entry, when $m \geq r$. In this case, $x^* = 1$. Geometrically, in this case for every $x > 1$, the line connecting $(x, G(mx))$ to $(0,0)$ has a larger slope than $G(mx)$.

Intuitively, if the firm reduces x – that is, increases entry – this has two effects on its profit. On the one hand, it increases the number of contestants who are paying for entry, which raises the profit. On the other hand, increasing entry makes it harder for each contestant that enters to win a prize. Hence, contestants' utility from entry decreases – consequently, the price that the firm can charge, expressed by (2), falls, lowering the profit. These two opposing effects determine the firm's optimal strategy.

At the same time, if the mass of prizes is sufficiently high relative to the mass of contestants, then each contestant who enters has a high chance of winning the prize. As a result, when additional contestants enter, this has only a small negative effect on the expected payoff from entry. Hence, increasing entry does not reduce contestants' willingness to pay for entry by a large amount. Thus, the negative effect of increasing entry on the firm's profit is outweighed by the positive effect at all levels of x. As a consequence, the optimal strategy involves full entry, that is, $x^* = 1$.

4 Comparative Statics

Proposition 1 describes how the equilibrium intensity of entry, expressed by x^*, is determined by the mass of prizes m, as well as by the value of the prize v. This section will analyse the effect of changes in these parameters on the equilibrium entry and the optimal price.

Recall that for a marginal contestant – the one whose type equals $\hat{\theta}$ – the expected payoff from entry equals $vG\left[\frac{m}{1-F(\hat{\theta})}\right]$, that is, the value of the prize times the probability of winning it. Thus, both v and m are positively related to the expected utility of entry.

As a benchmark, consider first what would happen if the price p was exogenously fixed at some level, as in standard models of contests with costly entry. If $vG(m) - p \geq 0$, then the payoff from entry would be positive even for a contestant with type 0, so all contestants would enter. Otherwise, entry would be determined by (2). An increase in v or m – that is, in the value of a prize, or the probability of winning it – would make entry more attractive for each contestant. Formally, an increase in v or m would increase the right-hand side of (2), and to restore equality, x would have to fall, corresponding to an increase in entry.

Here, however, the price is endogenously chosen by the firm. This means that it also adjusts at the equilibrium if m or v change. As we will see, changes in m and v affect the equilibrium in different ways.

As Proposition 1 shows, if the mass of prizes is sufficiently high – that is, if $m > r$ – then $x^* = 1$. A change in v has no effect at this corner solution. On the other hand, if $m < r$, then the equilibrium intensity of entry is given by (4). Since that condition does not depend on v, it follows that a change in v does not affect x^* at this equilibrium either. Formally, we have the following result:

Proposition 2. *At the equilibrium, an increase in v has no effect on x^*, and increases p.*

Thus, when the price of entry is endogenously set by a firm, an increase in v increases the marginal utility of entry, but this effect is absorbed by an increase in price, leaving the equilibrium intensity of entry unchanged. Intuitively, an increase in v raises the each contestant's payoff from entering the competition by the same factor. The firm can react by increasing the price by the same factor.

Now consider the effects of a change in m. An increase in m corresponds to a reduction in competitiveness, as the mass of prizes increases while the distribution of the mass of contestants remains unchanged. Unlike v, the mass of prizes enters the utility function nonlinearly. Consequently, the effect of an increase in m is different from that of an increase in v. In particular, an increase in m can change either the price or the intensity of entry, depending on the initial level of competitiveness. The following proposition describes this:

Proposition 3. *If $m < r$, then the optimal price does not change with m, while x^* is decreasing in m. If $m > r$, then the optimal price is increasing in m, while $x^* = 1$ does not change with m.*

In words, the effect of competitiveness on the equilibrium depends on its level. When the mass of prizes is low – specifically, when $m < r$ – the firm reacts to an increase in m by keeping the price constant while allowing the proportion of contestants who enter to increase. Thus, unlike an increase in v, an increase in m translates into greater intensity of entry but not into higher price. When the mass of prizes reaches r, full entry becomes the equilibrium. After that, a further increase in m induces the firm to raise the price, as in the case of an increase in v.

Intuitively, an increase in m, like an increase in v, raises each contestant's payoff from entering the competition. However, unlike an increase in v, it does not affect each contestant's utility in the same way. For a contestant with type θ, the probability of winning a prize after entering is $G\left(\frac{m}{1-F(\theta)}\right)$. An increase in m increases this probability, since it makes it more likely that the mass of contestants who enter and have higher types is below m. This increase is larger if G is steeper at $\frac{m}{1-F(\theta)}$. By Proposition 1, G is concave at $mx^* = \frac{m}{1-F(\hat{\theta})}$. Thus, around the threshold type $\hat{\theta}$, after an increase in m the payoff from entry rises faster for contestants with types below $\hat{\theta}$ – i.e. for those that previously were not entering the competition – than for those with types above $\hat{\theta}$. Hence, if $\hat{\theta} > 0$ – that is, when $x^* > 1$, which happens when $m > r$ – the firm can extract more rent by allowing some contestants with types below $\hat{\theta}$ to enter. Thus, increasing entry becomes optimal.

On the other hand, when $\hat{\theta} = 0$ – in other words, when $x^* = 1$, so all contestants enter – the firm cannot increase entry further. Its optimal response is then to raise the prize in a way that maintains full entry. Hence, the firm increases p while making sure that the contestant with type 0 remains indifferent between entering and not entering.

5 Extensions

Information available to the firm. One of the assumptions of the baseline model is that the firm has the same information as the contestants. Specifically, the firm is uninformed about the mass of contestants when it chooses the price of entry.

In some situations, however, it may be reasonable to think that the firm can have more information than the contestants. For example, the firm could be dealing with this situation over multiple periods, which could make it better informed.

Suppose that at the beginning of the game, the firm receives some signal σ about the mass of contestants y.[5] In principle, this could give rise to signalling equilibria in which the firm conditions the price on its information, and contestants update their beliefs about y after observing the price. At the same time, we can show that a pooling equilibrium identical to the one described previously would emerge even in this setup.

Consider a pooling strategy, in which the firm sets the same price for all realisations of σ. Then contestants do not update their beliefs about y after observing the equilibrium price. Hence, the best response of each contestant is the same as before – that is, a contestant enters if and only if her type is at least $\hat{\theta}$, where $\hat{\theta}$ is given by (4).

Then the firm's maximisation problem is

$$\max_{p\in[0,v]} \mathrm{E}\left[\frac{y}{x}p \mid \sigma\right] \text{ subject to } p = vG\left(mx\right)$$

where x is defined as in Section 3. The only difference is that the firm is now informed about y. Rewriting the maximisation problem in terms of x, we can write it analogously to (3) as

$$\max_{x\in[1,+\infty)} \frac{vG\left(mx\right)}{x}\mathrm{E}\left[y \mid \sigma\right]$$

This has the same solution as (3). As before, the optimal x, and hence the equilibrium price, does not depend on y. Thus, the pooling strategy, in

[5] A special case of this is when the firm is fully informed about y.

which the firm selects the same price for all values of y, is a best response for the firm, as long as we define sufficiently strong off-equilibrium beliefs for contestants – for example, if contestants believe that $y \to \infty$ if they observe a price different from the equilibrium price. Hence, a pooling equilibrium exists, which has the same properties as the one found in Section 3. In particular, the comparative statics derived in Section 4 continue to hold.

Generic G. The baseline model assumed that G is unimodal. Recall that the firm's problem, given by (3), is equivalent to maximising the average value of the function $G(mx)$. When G is unimodal, this problem has a unique solution, in which mx lies on a concave part of G.

When G has a generic shape, the firm's problem is unchanged. Hence, any interior solution x^* is still pinned down by a first-order condition in such a way that mx^* is located on a weakly concave section of G. To see this, note that if G is convex at mx, the firm can increase the average value of $G(mx)$ by increasing x.

At the same time, when G has multiple modes, Lemma 2 ceases to hold, because there may now be multiple values of r for which $g(r) = \frac{G(r)}{r}$. Let $Z \equiv \left\{ r : g(r) = \frac{G(r)}{r} \right\}$ be the set of points at which the slope of G equals the average value of G. At any interior equilibrium, x^* must be such that $mx^* \in Z$. More formally, we have the following result:

Lemma 3. *Suppose that G has a generic shape. At any equilibrium, either $x^* = 1$, or $mx^* \in Z$.*

Hence, a modified version of Proposition 1 holds when G is multimodal: either $x^* = 1$, or x^* is such that the slope of G equals the average value of G at mx^*. The firm's problem, given by (3), can then be written as

$$\max_{x \in [1,+\infty)} \frac{vG(mx)}{x} \mathrm{E}[y] = \max_{x \in [1,+\infty)} \frac{G(mx)}{mx} = \max_{z \in Z \cup \{m\}} \frac{G(z)}{z}$$

where the last equality substitutes $z \equiv mx$, using Lemma 3 and the fact that maximising with respect to mx is equivalent to maximising with respect to x.

For some values of m and shapes of G, this maximisation problem can have multiple solutions – referring to Figure 1, the line connecting $(x, G(mx))$ to $(0,0)$ can be tangent to $G(mx)$ at multiple points. In this case, multiple equilibria exist. However, if the equilibrium is unique, the result summarised in 2 remains true:

Proposition 4. *Suppose G has a generic shape, and the equilibrium is unique. An increase in v has no effect on x^*, and increases p.*

Thus, it remains true that an increase in v raises the optimal price and has no effect on the intensity of entry.

Furthermore, if the equilibrium is unique, a result similar to the one described in Proposition 3 emerges:

Proposition 5. *Suppose G has a generic shape, and the equilibrium is unique. If $x^* = 1$, then the optimal price is increasing in m, while x^* does not change with m. If $x^* > 1$, then the optimal price does not change with m, while x^* is decreasing in m.*

6 Conclusions

Entry into competitive settings, such as auctions or contests, has been a focus of much research. This paper considers situations in which access to the competition is controlled by a profit-maximising firm. The paper looks at how the value of a prize and the mass of prizes affect the price that the firm charges for entry, and the equilibrium intensity of entry.

I find that an increase in the value of a prize leads the firm to increase the price, but the intensity of entry remains constant. On the other hand, the reaction of the firm to an increase in the mass of prizes depends on its initial level. When the mass increases from a small level, the firm responds by increasing the intensity of entry while keeping the price unchanged. At some point entry reaches the maximum level, and a further increase in the mass of prizes leads the firm to raise the price of entry.

One particular feature of the model is that the firm is a monopolist. This corresponds to examples such as association membership, or GRE test in university admission. Future research can look at situations in which there are two or more channels of entering the competition (for example, several different university admission tests), each controlled by a different firm.

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Appendix

Proof of Lemma 1. If a contestant with type θ enters, her expected payoff is

$$v \Pr\left[y \int_{\theta}^{1} h(t)\, dF(t) \leq m\right] - p = vG\left[\frac{m}{\int_{\theta}^{1} h(t)\, dF(t)}\right] - p \equiv w(\theta)$$

which is weakly increasing in θ. Note that $w(\theta) \geq 0$ for all types for which $h(\theta) > 0$, and $w(\theta) \leq 0$ for all types for which $h(\theta) < 1$. Denote $\hat{\theta} \equiv \sup\{\theta \mid h(\theta) < 1\}$, that is, the highest type that does not enter with certainty.

Suppose first that $\hat{\theta} = 0$, so all contestants enter. Then $\int_{\hat{\theta}}^{1} h(t)\, dF(t) = 1$, so $w\left(\hat{\theta}\right) = vG(m) - p$. If $w(0) > 0$, then, since $w(\theta)$ is weakly increasing, he have $w(\theta) > 0, \forall\theta$. Hence, the firm can increase p while ensuring that $w(\theta)$ remains positive for all θ. Thus, the firm can raise its profit by deviating.

Hence, at the equilibrium we must have $w\left(\hat{\theta}\right) = w(0) = vG(m) - p = 0$, so (1) holds.

Suppose instead that $\hat{\theta} > 0$. Take some $\theta' < \hat{\theta}$. If $h(\theta') > 0$, then $w(\theta') \geq 0$. But $w\left(\hat{\theta}\right) \leq 0$, and $w(\cdot)$ is strictly increasing over any interval over which $h(\cdot) > 0$, so we must have $h(t) = 0$ for almost all $t \in \left(\theta', \hat{\theta}\right)$. Thus, $h(\theta) = 1$ for all $\theta > \hat{\theta}$, and $h(\theta) = 0$ for almost all $\theta < \hat{\theta}$, hence $\int_0^{\hat{\theta}} h(\theta)\, dF(\theta) = 0$. Thus, $\hat{\theta}$ is given by $vG\left[\frac{m}{\int_{\hat{\theta}}^1 dF(t)}\right] = p$, which is equivalent to (1). □

Proof of Lemma 2. First, note that $g(x)$ is increasing in x for all $x < k$. Hence, for all $x \leq k$ we have $G(x) = \int_0^x g(z)\, dz < \int_0^x g(x)\, dz = xg(x)$. Thus, $G(x) - xg(x) < 0$ for all $x \leq k$. On the other hand, when $x > k$, we have $\frac{d}{dx}[G(x) - xg(x)] = -xg'(x) > 0$, where the inequality comes from the fact that $g(x)$ is decreasing in x for all $x > k$. Furthermore, since g is a density, $\lim_{x\to\infty} xg(x) = 0$, and hence $\lim_{x\to\infty}[G(x) - xg(x)] = 1 > 0$. Hence, for $x > k$, the expression $G(x) - xg(x)$ is continuous, strictly increasing, negative for sufficiently small x, and positive for sufficiently large x. This implies that there is a unique $r \in (k, +\infty)$ such that $G(x) - xg(x) > 0$ if and only if $x > r$. □

Proof of Proposition 1. Let $\pi(x) = \frac{vG(mx)}{x}\mathrm{E}[y]$ be the firm's profit. Note that

$$\pi'(x) = v\frac{mxg(mx) - G(mx)}{x^2}\mathrm{E}[y] \tag{5}$$

To prove the first statement, suppose that $m \geq r$. By Lemma 2, this implies that $mg(m) - G(m) \leq 0$ and $m > k$. The former, together with (5), implies that

$$\pi'(1) = v[mg(m) - G(m)]\,\mathrm{E}[y] \leq 0$$

The latter implies that for all $x \geq 1$, $g'(mx) < 0$, and thus

$$\frac{d}{dx}[mxg(mx) - G(mx)] = mg(mx) + m^2xg'(mx) - mg(mx) = m^2xg'(mx) < 0$$

Hence, $mxg(mx) - G(mx)$ is weakly negative at $x = 1$, and strictly decreasing in x for all $x \geq 1$. Thus, for all $x > 1$, we have $mxg(mx) -$

$G(mx) < 0$. By (5), this implies that $\pi'(x) < 0$ for all $x > 1$. Hence, $\pi(x)$ is decreasing in x for all $x \geq 1$, and thus $x^* = 1$.

To prove the second statement, note that if $m < r$, then by Lemma 2, $G(m) - mg(m) < 0$. Hence, $\pi'(1) = v[mg(m) - G(m)]\,\mathrm{E}[y] > 0$. Thus, profit is increasing in x at $x = 1$, and hence $x^* > 1$. Since x^* is in the interior of $(1, +\infty)$, it is given by the first-order condition $\pi'(x^*) = 0$. Equating (5) to zero at $x = x^*$ yields

$$mx^* g(mx^*) - G(mx^*) = 0$$

which implies that $mx^* = r$. □

Proof of Proposition 2. At the equilibrium, x^* is given by (3), and can be written as

$$x^* = \arg\max_{x\in[1,+\infty)} \frac{vG(mx)}{x}\mathrm{E}[y] = \arg\max_{x\in[1,+\infty)} \frac{G(mx)}{x}$$

which does not depend on v. At the same time, the optimal price is given by (2) as $p = vG(mx^*)$, which increases in v. □

Proof of Proposition 3. If $m > r$, then, by Proposition 1, $x^* = 1$, and hence the optimal price equals $vG(m)$. An increase or decrease in m that leaves the $m > r$ inequality unchanged does not affect x^* while, respectively, increasing or decreasing the optimal price.

If $m < r$, then, by Proposition 1, $x^* = \frac{r}{m}$. This decreases in m. At the same time, the optimal price equals $vG(r)$, which is constant in m.

Proof of Lemma 3. Let $\pi(x) = \frac{vG(mx)}{x}\mathrm{E}[y]$. Take any equilibrium at which $x^* > 1$. Then x^* must satisfy the first-order condition

$$\pi'(x^*) = v\frac{mx^* g(mx^*) - G(mx^*)}{(x^*)^2}\mathrm{E}[y] = 0$$

Hence, if $x^* > 1$, then $mx^* g(mx^*) - G(mx^*) = 0$, so $g(mx^*) = \frac{G(mx^*)}{mx^*}$. Thus, either $x^* = 1$, or $mx^* \in Z$. □

Proof of Proposition 4. Identical to the Proof of Proposition 2. □

Proof of Proposition 5. If $x^* = 1$ and the equilibrium is unique, then a sufficiently small change in m has no effect on x^*. The optimal price is then given by (2) as $vG(m)$, which is increasing in m.

If $x^* > 1$, then by Lemma 3 it is given by $x^* = \frac{r}{m}$ for some $r \in Z$. An increase in m decreases x^*. The optimal price is then given by (2) as $vG(mx^*) = vG(r)$, which is constant in m. $\square$